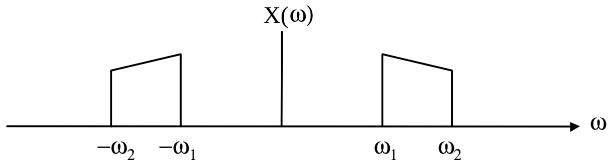
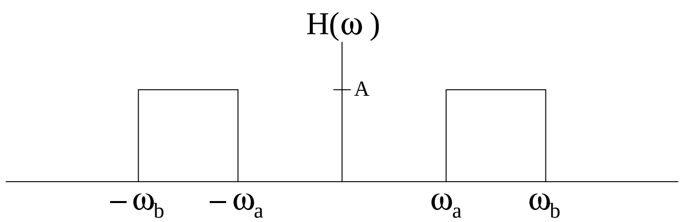
Sampling Exercises

1. The sampling theorem as we have introduced it states that a signal *x*(*t*) must be sampled at a rate greater than its highest frequency. This implies that if *x*(*t*) has a spectrum as indicated in the figure below, then *x*(*t*) must be sampled at a rate greater than $2w_2$. However, if *x*(*t*) is a *bandpass* signal, it may be sampled at a rate which is lower than twice the highest frequency present. There are a variety of techniques for sampling such signals, and these techniques are generally referred to as *bandpass sampling.*

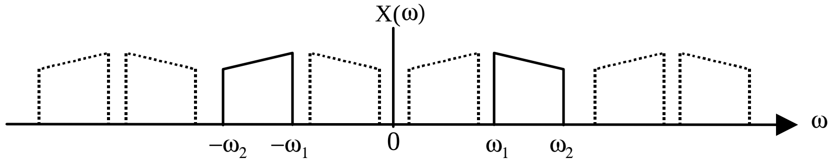


Let *x*(*t*) be sampled using an impulse train $p(t) = \sum_{n=-\infty}^{+\infty} d(t - nT)$, with sampling interval *T*, and recovered using an ideal bandpass filter having the gain A and frequency response as shown below.



Assuming that $w_1 > (w_2 - w_1)$, find the maximum value of *T* and the values of the constants A, w_a, and w_b such that the recovered signal $x_r(t) = x(t)$. (Hint: You will find it helpful to graphically depict the aliasing which results due to sampling. Then determine the lowest sampling interval for which the aliasing will be eliminated by the bandpass filters.)

Solution: A sampling frequency $w_s > 2w_2$ is sufficient to prevent aliasing. If the sampling frequency is lowered into the interval $2w_1 \le w_s \le 2w_2$, there will be overlapping of bands (i.e., the band between $-w_2$ and $-w_1$ will be aliased into an overlap with the band between w_1 and w_2). If the sampling frequency drops further into the range $w_2 < w_s < 2w_1$, then there is no longer any overlapping (see illustration).



Thus, the lower bound on the sampling frequency is w_2, and the upper bound on the sampling interval is $T < \frac{2p}{w_2}$. If we can consider the magnitude of $X(w)$ at $\pm w_2$ to be zero, then the sampling interval may be exactly $T = \frac{2p}{w_2}$. If the magnitude of $X(w)$ at $\pm w_2$ is non-zero, then the maximum sampling interval is $T = \frac{2p}{w_2} - e$, where ε is vanishingly close to zero.

To obtain a recovered signal $x_r(t) = x(t)$, the gain of the bandpass filter must counteract the $1/T$ attenuation factor due to sampling. Thus, the gain of the bandpass filter must be $A = T$. The bandpass filter must pass the original spectrum while rejecting the copies. This requires that $w_s - w_1 < w_a \le w_1$ and $w_2 \le w_b < 2w_s - w_2$, where $w_2 < w_s < 2w_1$ (or $w_2 \le w_s < 2w_1$ if $X(w_2) = X(-w_2) = 0$).

2. In a linear time invariant system, two signals $x_1(t)$ and $x_2(t)$ are multiplied to give an output signal $y(t) = x_1(t)x_2(t)$, which is then sampled using an impulse train $p(t) = \sum_{n=-\infty}^{+\infty} d(t - nT)$, with sampling interval *T*. $x_1(t)$ is bandlimited to w_1 and $x_2(t)$ is bandlimited to w_2, that is:

$$X_1(w) = 0, \quad |w| > w_1$$
$$X_2(w) = 0, \quad |w| > w_2$$

Determine the maximum sampling interval *T*, such that *y*(*t*) is recoverable from the sampled signal through the use of an ideal lowpass filter.

Solution:

$$\begin{aligned} Y(w) &= \frac{1}{2p}\left[X_1(w) * X_2(w)\right] \\ &= \frac{1}{2p}\int_{-\infty}^{\infty} X_1(q)X_2(w-q)\,dq \\ &= \frac{1}{2p}\int_{-w_1}^{w_1} X_1(q)X_2(w-q)\,dq \end{aligned}$$

The integral $Y(w)$ must be zero wherever:

$$X_2(w-q) = 0\ , \quad |w-q| > w_2$$

with q ranging from $-w_1$ to $+w_1$. Considering the boundary points of integration:

For $q = -w_1$: $X_2(w + w_1) = 0$, $|w + w_1| > w_2 \Rightarrow w > -w_1 + w_2$ or $w < -w_1 - w_2$

For $q = w_1$: $X_2(w - w_1) = 0$, $|w - w_1| > w_2 \Rightarrow w > w_1 + w_2$ or $w < w_1 - w_2$

Thus, over the range of integration:

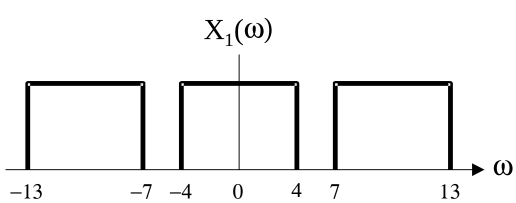
$$Y(w) = 0 \quad \text{for} \quad |w| > w_1 + w_2$$

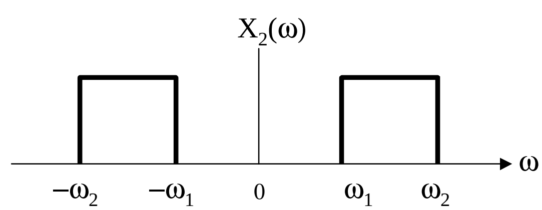
Sampling requirements:

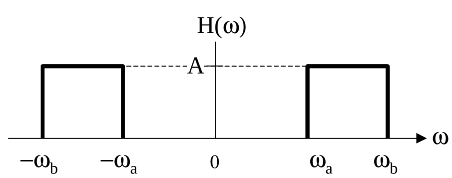
$$w_s > 2(w_1 + w_2)$$

$$T = \frac{1}{f_s} = \frac{2p}{w_s} \qquad T < \frac{p}{w_1 + w_2}$$

3. In a linear time invariant system, two signals $x_1(t)$ and $x_2(t)$ are convolved to give an output signal $y(t) = x_1(t) * x_2(t)$. The signal $x_1(t)$ has the periodic spectrum $X_1(w)$ and the signal $x_2(t)$ has In a linear time invariant system, two signals $x_1(t)$ and $x_2(t)$ are convolved to give an output signal $y(t) = x_1(t) * x_2(t)$. The signal $x_1(t)$ has the spectrum $X_1(w)$ and the signal $x_2(t)$ has the spectrum $X_2(w)$, as indicated below (these drawings are not to scale). Let the signal $y(t)$ be sampled using an impulse train $p(t) = \sum_{n=-\infty}^{+\infty} d(t - nT)$, with sampling interval T, and then recovered using an ideal filter having the gain A and frequency response $H(w)$ as shown below.

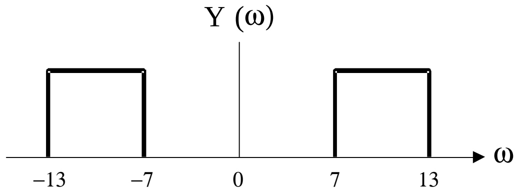




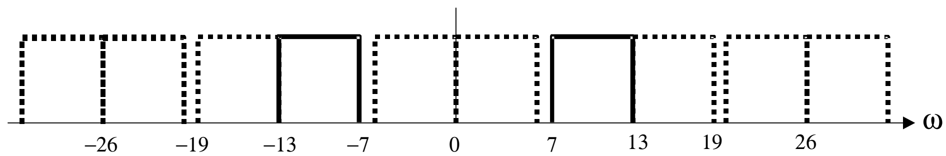


a) Assuming that $w_1 = 6$ and $w_2 = 14$, find the maximum value of T and the values of the constants A, w_a, and w_b such that the recovered signal $y_r(t) = y(t)$.

Solution: $Y(w) = X_1(w)X_2(w)$



For this spectrum, bandpass sampling is possible because the $7 > 13 - 7$. This leaves enough room for the periodic replicas of the spectra produced by sampling to be interleaved as indicated below:



Interleaving will work for $13 < w_s < 14$. Thus, we have:

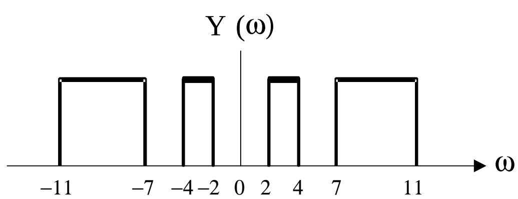
$$\frac{2p}{14} < T < \frac{2p}{13} = .4833 \qquad Max\ T = .4833 \quad A = T = .4833$$

Our bandpass filter must pass the spectrum between 7 and 13, while rejecting the replicas:

$$w_b = 13 \quad 6 < w_a \leq 7$$

b) Assuming that $w_1 = 2$ and $w_2 = 11$, find the maximum value of T and the values of the constants A, w_a, and w_b such that the recovered signal $y_r(t) = y(t)$.

Solution: $Y(w) = X_1(w)X_2(w)$



The sampling process produces periodic replicas of the spectrum at intervals of w_s. For this spectrum, bandpass sampling cannot be used because the replicas overlap (causing aliasing) for any sampling frequency $w_s \leq 22$. There are numerous overlaps that occur for sampling rates up to 22. Consider:

- Sampling rates less than 2 will cause an aliased replica of the 2 to 4 portion of the spectrum to overlap itself.
- Sampling rates less than 4 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap itself.
- Sampling rates from 1.5 to 4.5 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap an aliased replica of the 2 to 4 portion of the spectrum.
- Sampling rates from 2 to 4 will cause an aliased replica of the 2 to 4 portion of the spectrum to overlap an aliased replica of the –4 to –2 portion of the spectrum.
- Sampling rates from 3 to 9 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap the 2 to 4 portion of the spectrum.
- Sampling rates from 4 to 8 will cause an aliased replica of the 2 to 4 portion of the spectrum to overlap the –4 to –2 portion of the spectrum.
- Sampling rates from 4.5 to 7.5 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap an aliased replica of the –4 to –2 portion of the spectrum.
- Sampling rates from 7 to 11 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap an aliased replica of the –11 to –7 portion of the spectrum.
- Sampling rates from 9 to 15 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap the –4 to –2 portion of the spectrum.
- Sampling rates from 14 to 22 will cause an aliased replica of the 7 to 11 portion of the spectrum to overlap the –11 to –7 portion of the spectrum.
- These are only some of the overlaps that will occur.

Thus, to avoid aliasing, we must use baseband sampling:

$$w_s > 22 \qquad T < \frac{2p}{22} = .2856 \qquad Max\ T = .2856 - e \qquad A = T = .2856$$

Our bandpass filter must pass the spectrum between 2 and 11, while rejecting the replicas:

$$w_b = 11 \quad 0 \le w_a \le 2$$